P360: Physical Optics

Supplementary Note # 4: Using Matrices for Geometric Optics

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Geometric Optics Using Matrices

Some Examples

Example 1

Consider an object which is focused on a screen a distance D away using a converging lens of focal length f. The distance between the lens and screen is x and between the object and lens is $D - x$. So, we have drift space of length $D - x$, followed by lens, followed by drift space of length x. The overall matrix is the product of three matrices:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & D - x \\ 0 & 1 \end{pmatrix}$$

The result is:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 - \frac{x}{f} & \frac{D(f - x) - x^2}{f} \\ -\frac{1}{f} & \frac{-D + f + x}{f} \end{pmatrix}$$

The condition for object to image focus is met when the element B vanishes. Solving for the quadratic equation:

$$\frac{D(f - x) - x^2}{f} = 0$$

yields two solutions, whose difference d is given by:

$$d = \sqrt{D(D - 4f)}$$

So there are two positions of the lens which will lead to a focus, separated by the above distance. Note also that this tells us that the minimum distance, D, between the object and screen for obtaining a focus is 4f - else d is imaginary. This leads us to the so-called Bessel method for finding the focal length of a lens - one determines the two positions which lead to a focus for fixed D and measures the difference d, then:

$$f = \frac{D^2 - d^2}{4D}$$

Example 2

Consider a beam broadener: A converging lens of focal length f_1 followed by another of focal length f_2. Suppose the lens are separated by distance $f_1 + f_2$. Again, the overall matrix is a sum product of three matrices:

$$\begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & f_1 + f_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix}$$

resulting in:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -\frac{f_2}{f_1} & f_1 + f_2 \\ 0 & -\frac{f_1}{f_2} \end{pmatrix}$$

An incident parallel beam of width W_1 turns into an outgoing parallel beam of width W_2 where:

$$W_2 = \frac{f_2}{f_1} W_1$$

Note that the final matrix has element A (magnification) consistent with the above and element C is zero, consistent with an overall focal length of infinity for this system.

This setup will be used to ‘clean up’ a laser beam and tune the spot size. Place an iris at the point between the two lenses where the beam comes to a focus.

Example 3

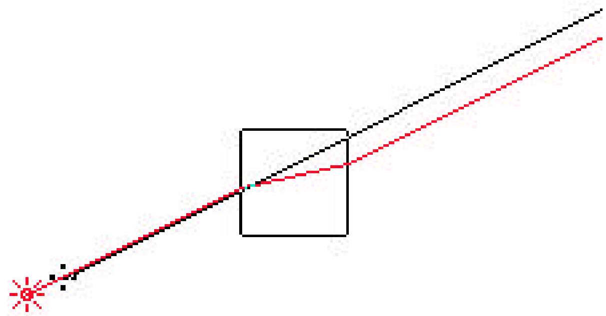
Consider a slab of thickness L and index of refraction n. The matrix for this can be represented as a product of three matrices: two for the interfaces and one for a drift space of length L:

$$\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}$$

resulting in:

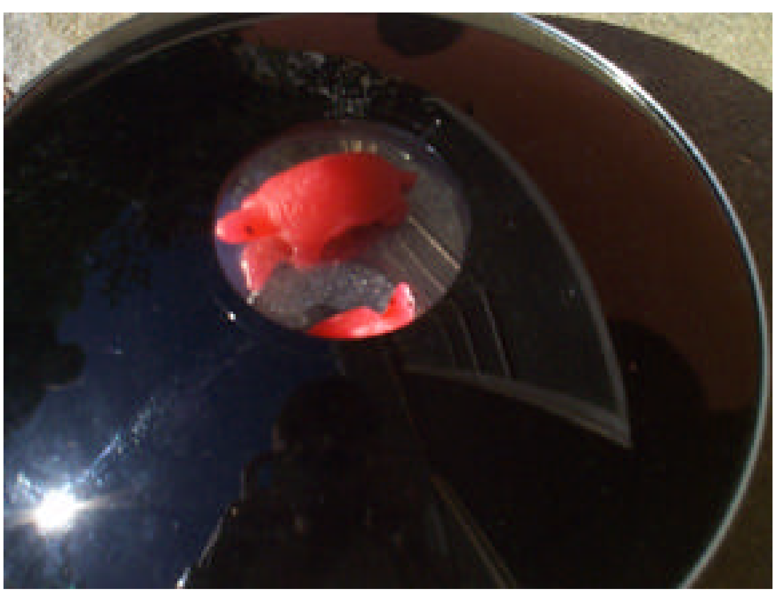
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & L/n \\ 0 & 1 \end{pmatrix}$$

Does this make sense ? Consider a parallel beam entering the slab perpendicular to the slab. What about the emergent beam - still parallel ? This corresponds to a focal length of infinity. Suppose the incident beam is now slightly tilted. After the beam emerges the what is the displacement of the beam? See the figure below.



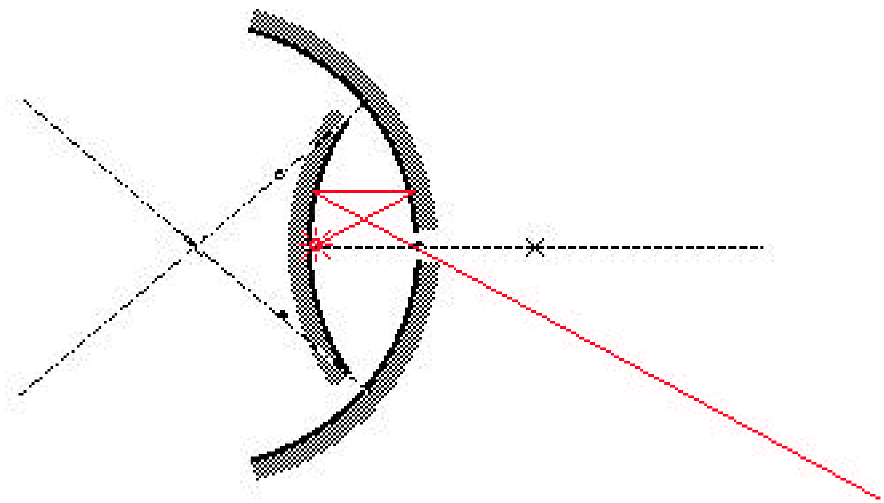
Example 4

Mirage Mirror. This setup was described in the lab on the mirage toy. Two spherical mirrors of same focal length face each other. One of the mirrors (top) has a hole in the center. The spacing between the mirrors is just the focal length. An object placed at the center of the bottom mirror has a real inverted image at the hole of the top mirror. When you see it you want to touch it.



Mirage Toy: Here is a photo of the mirage toy. A plastic toy turtle is on the bottom mirror and you can see its inverted image at the hole of the top mirror.

The figure below shows some representative rays. It is not too hard to see what is going on. A ray from the object starts at the focus of the top mirror and is reflected parallel to the axis towards the bottom mirror where it is reflected to the focal point of the bottom mirror which is at the hole of the top mirror.



Let's do this with matrices. The matrix for a concave mirror is:

$$\begin{pmatrix} 1 & 0 \\ 2/r & 1 \end{pmatrix}$$

Notice the sign convention and recall that for a mirror the focal length is just the half of the radius of the mirror. The matrix for the drift space of length d is:

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

We are letting the space between the mirrors be a variable. A ray starting from the object goes through drift space, reflects, drifts, reflects, and drifts. The overall matrix is a product of 5 matrices:

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2/r & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2/r & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

which when multiplied yields:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \dfrac{4d^2 + 6dr + r^2}{r^2} & \dfrac{d\left(4d^2 + 8dr + 3r^2\right)}{r^2} \\ \dfrac{4(d+r)}{r^2} & \dfrac{4d^2 + 6dr + r^2}{r^2} \end{pmatrix}$$

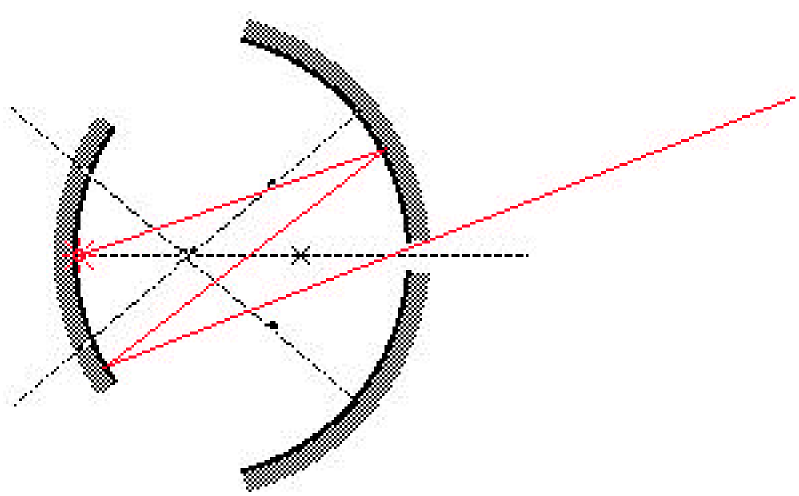
The condition for focusing requires that the element B vanish which means solving the quadratic:

$$4d^2 + 8dr + 3r^2 = 0$$

The solutions are $d = -r/2$ and $d = -3r/2$. So the first one is not unexpected but the second one is not obvious. If you subsitute the first solution into the expression for element A:

$$A = \frac{4d^2 + 6dr + r^2}{r^2}$$

you find that $A = -1$. Recall that if B = 0, then A is the magnification. So this tells us that the image is inverted. If you substitute the second solution for d you find $A = +1$. In fact you can slowly lift the top mirror of the toy and see the uninverted image pop up when the separation is 3 times the focal length. See the simulation below using the Ray program.



Note: I did these matrix calculations using *Mathematica* which makes it a lot easier to handle those messy matrix calculations. So now we can really 'go to town.' So far we considered 3 drifts and 2 reflections. The next level of complication would be 5 drifts and 4 reflections which means multiplying 9 matrices. Here's the result:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \dfrac{16d^4 + 56d^3r + 60d^2r^2 + 20dr^3 + r^4}{r^4} & \dfrac{d\left(16d^4 + 64d^3r + 84d^2r^2 + 40dr^3 + 5r^4\right)}{r^4} \\ \dfrac{8\left(2d^3 + 6d^2r + 5dr^2 + r^3\right)}{r^4} & \dfrac{16d^4 + 56d^3r + 60d^2r^2 + 20dr^3 + r^4}{r^4} \end{pmatrix}$$

The condition that B = 0 involves solving for the roots of:

$$16d^4 + 64d^3r + 84d^2r^2 + 40dr^3 + 5r^4 = 0$$

which are:

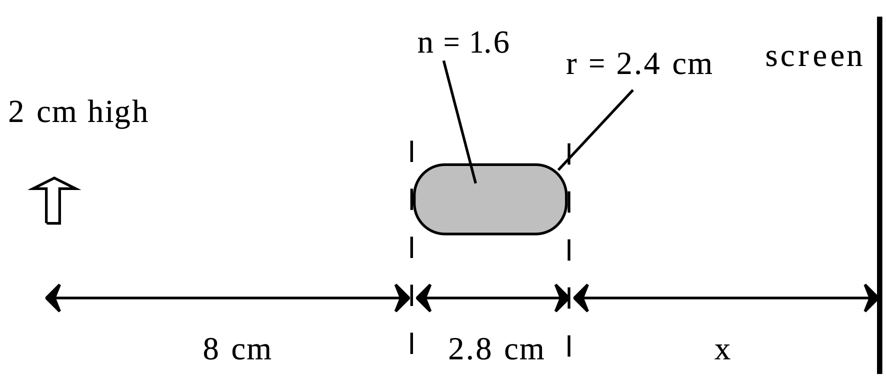
$$d = \frac{1}{4}\left(-5r - \sqrt{5}r\right)$$
$$d = \frac{1}{4}\left(-5r + \sqrt{5}r\right)$$
$$d = \frac{1}{4}\left(-3r - \sqrt{5}r\right)$$
$$d = \frac{1}{4}\left(-3r + \sqrt{5}r\right)$$

and substitution into the expression for the element A will yield the magnifications.

Problem 1

Consider the arrangement shown below of an object, glass rod and image. The object is 2 cm tall and located 8 cm to the left of the rod. The object is imaged on a screen which is distance x from the other end of the glass rod. Find x and the size of the image.

Note that the ends of the glass rod are spherical with radii of curvature equal to 2.4 cm. The length of the rod is 2.8 cm and the index of refraction of the glass is 1.6.



Problem 2

A converging lens (f = +8 cm) is 6 cm to the left of a diverging lens (f = -12 cm). A 3 cm tall object is on-axis and located 24 cm to the left of the converging lens. The object is imaged a distance x away to the right of the diverging lens. Find x and the size of the image.

Problem 3

An old lantern slide is 2 inches tall. We want to use a converging lens to image the slide on a screen which is 10.5 feet away from the slide. If the image is to be 40 inches on the screen, what is the focal length of the lens and where should it be placed relative to the slide and screen ?